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Ideal and Resistive Magnetohydrodynamic Modes

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This paper gives a further discussion of the analytical properties of both discrete and continuous Alfven wave spectra in an incompressible as well as compressible plasma. Although the continuous MHD modes produced by a well-behaved initial perturbation decay according to a power law, some singular solutions exist and are found to behave differently. In particular, it is possible to exhibit the existence of a new continuous mode which decays exponentially, and not as an inverse power of time, and this exponential damping is not the consequence of a continuous variation of the magnetic field. Even the set of discrete magnetohydrodynamic modes is shown to be empty unless certain conditions are satisfied. Next, we consider resistive modes and give explicit solutions for them which are valid in the neighborhood of the Alfven resonance layer and discuss their implications for plasma heating schemes. Finally, we study discrete and continuous Alfven wave spectra in a compressible plasma and discuss how they behave differently from those in an incompressible plasma. In particular, we show that though compressibility of the plasma is responsible for the slow mode continuum, strong compressibility will eliminate it. The discrete modes in a compressible plasma undergo an exponential damping even in an ideal plasma if the compressibility is strong.

1. INTRODUCTION

The propagation of Alfven waves in a nonuniform plasma has been a problem of much interest because of the use of Alfven waves for radio-frequency heating of a fusion plasma (Chen and Hasegawa, 1974) and the solar corona (Ionson, 1978).

For an infinite, homogeneous plasma in an inhomogeneous magnetic field $B_0(x)$, the governing equation becomes singular (Velikhov, 1962; Uberoi, 1972) for resonant frequencies ω which correspond to the Alfven modes and the spectrum for the branch is a continuum. This problem is analogous to that of longitudinal electrostatic oscillations in an inhomogeneous plasma considered by Barston (1964), and one may transfer many of

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Barston's results straightaway to the present problem. Thus, if the magnetic field intensity is monotonic, the continuous spectrum is simple, though the associated eigenfunctions $v(x)$ are then singular; one can construct a well-behaved solution of the original initial-value problem by integrally superimposing these eigenfunctions over the whole spectrum, that is,

$$V_k(x, t) = \int_s A_k(\omega) V_{k\omega}(x)\, e^{-i\omega t}\, d\omega$$

Further, we have an asymptotic estimate of $V_k(x, t) \sim (1/t) \exp[\pm ikV_A(x)t]$, where k is the wavenumber of the modes, and $V_A^2 = B_0^2/\rho$. Thus, each infinitesimally thin plasma layer supports Alfven waves corresponding to the local magnetic field intensity $B_0(x)$, and these Alfven modes are completely out of coordination and hence are damped as the inverse power of time. Hence, if a surface Alfven wave is excited by an external coupler, the wave will be phase mixed by the Alfven resonance, and its energy can be dissipated in the plasma. However, some singular solutions exist which behave differently. It is possible to exhibit, in particular, the existence of a new continuous mode which decays exponentially, and not as an inverse power of time, as we will show in this paper.

On the other hand, if B_0 is constant, say C, or jumps from one constant value B_{01} to another constant value B_{02} $(>B_{01})$ in some open interval I, then that interval contributes to the discrete spectrum those values of ω such that $\omega = \pm C/\sqrt{\rho}$ or $kV_{A1} < \omega < kV_{A2}$. The corresponding eigenfunctions are well-behaved. Sedlacek (1971) considered an initial value for small-amplitude electrostatic oscillations in an inhomogeneous cold plasma. Following Sedlacek's analysis, one may consider what happens to the above discrete spectrum when the jump in the magnetic field intensity profile is smoothed. This was done by Tataronis and Grossman (1973) and Tataronis (1975), who exploited the formal identity of the equations describing electrostatic oscillations in an inhomogeneous plasma and the Alfven wave propagation in an inhomogeneous medium and translated the results of Sedlacek (1971) into the MHD domain. Thus, it was found that, in the limit of small smoothing of the magnetic field, these discrete modes undergo an exponential damping due to poles on the nonprincipal Riemann sheets. Considerable controversy has apparently resulted in the literature regarding the implications of this result. Lee (1980) and later Cally (1991) pointed out that such decaying modes are not normal modes of the system (because the ideal MHD constitutes a Hermitian system and can support only real eigenvalues), and their decay rate cannot be interpreted as a dissipation rate, so that no physical significance can be attached to them. However, Lee and Roberts (1986) argued that this decay rate can be viewed as that due to mode conversion of the discrete modes into local oscillations within the smoothed resonant layer, while Steinolfson (1985) and Hollweg (1987*a*) argued that this decay rate is due to a vanishingly small dissipation in conjunction with cascading of energy to progressively smaller spatial scales due to phase mixing. Cally (1991) pointed out that the phase mixing is no consequence of the continuous variation of the field profile.

The inclusion of additional physical effects such as finite-ion-Larmor-radius effects (Hasegawa and Chen, 1976) or resistivity η (Davies, 1984; Lortz and Spies, 1984; Ryu and Grimm, 1984; Pao and Kerner, 1985; Mok and Einaudi, 1985; Riedel, 1986; Poedts and Kerner, 1991) increases the order of the governing differential equation and eliminates the singularity associated with the Alfven resonance and hence the continuous spectrum. This leads to discrete modes which then decay exponentially and the corresponding eigenfunctions are localized and oscillatory in the resonant layer (Poedts and Kerner, 1991). Pao and Kerner (1985) considered the solvability of the eigenvalue problem for the resistive modes and its relation to the anti-Stokes lines for the problem. The resistive eigenvalue problem was shown to have no solution when two anti-Stokes lines cross the basic interval and one has two resistive layers on the latter even in the ideal limit. [This situation is similar to the one produced by viscous effects in hydrodynamic stability (Lin, 1957).] Mok and Einaudi (1985) studied the resistive modes (in the limit of small but finite resistivity) by using a boundary layer technique and claimed that there is a mode-damping due to resistivity. But since the damping rate found by Mok and Einaudi (1985) was independent of resistivity when it is small, as Hollweg (1987*b*) pointed out, the mode damping is actually that due to resonance absorption. The role of resistivity is merely to remove the mathematical singularity (Y. Mok, personal communication, 1986). Steinolfson (1985), however, argued that the damping rate (even to lowest order) should be dependent on resistivity when resistive effects are present, no matter how small. Ryu and Grimm (1984) and Reidel (1986) found that, in the presence of resistivity, the continuous spectrum is replaced by point eigenvalues on specific curves in the stable ω half-plane which become independent of resistivity in the limit of zero resistivity. The damping of the resistive normal mode, in particular, remains finite in limit $\eta \Rightarrow 0$ (Poedts and Kerner, 1991), so that these resistive normal modes do not correspond to the ideal MHD eigenmodes. Explicit solutions of the resistive modes valid in the neighborhood of the Alfven resonance layer have not apparently been given so far. We will address this issue in this paper.

In a compressible plasma, the conditions for the existence of a continuous spectrum have not yet been clarified and existing accounts are mostly conjectural (Chen and Hasegawa, 1974; Grad, 1973; Goedbloed, 1983; Hameiri, 1985; Hollweg and Yang, 1988). In this paper, we will show that the slow mode continuous spectrum exists only if the compressibility effects are weak. But strong compressibility are found to eliminate the continuous spectrum like the nonideal effects such as resistivity. We will show that even the discrete spectra behave differently in a compressible plasma—they undergo an exponential damping even in an ideal plasma if the compressibility is strong!

In this paper, we adopt a plane slab geometry like many others previously; though it is not directly relevant to fusion devices, it affords additional insights.

2. DISCRETE MHD SPECTRUM IN AN INCOMPRESSIBLE PLASMA

The MHD equations for an ideal incompressible plasma are

$$\nabla \cdot \mathbf{V} = 0 \tag{1}$$

$$\rho \frac{D\mathbf{V}}{Dt} = -\nabla p + (\nabla \times \mathbf{B}) \times \mathbf{B} \tag{2}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) \tag{3}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{4}$$

We have taken the density to be uniform so that the effect of magnetic field inhomogeneity on the Alfven spectrum can be understood more clearly.

We consider free oscillations in a plasma infinite in the y and z directions and contained between two ideally conducting plates at $x = x_1$ and $x = x_2$ and subjected to a nonuniform magnetic field $\mathbf{B}_0 = B_0(x)\hat{\mathbf{i}}_y$; the equilibrium state varies only in the x direction and is given by

$$\frac{d}{dx}\left(p_0 + \frac{1}{2} B_0^2\right) = 0 \tag{5}$$

For linearized perturbations, we obtain from equations (1)–(4)

$$\nabla \cdot \left[\left(\frac{\partial^2}{\partial t^2} - \frac{(\mathbf{B}_0 \cdot \nabla)^2}{\rho}\right) \nabla V_x\right] = 0 \tag{6}$$

On making a normal-mode analysis with Fourier decomposition in y and t according to

$$V_x \sim V_x(x)\, e^{i(ky - \omega t)} \tag{7}$$

it turns out that $V_x(x)$ satisfies the equation

$$\frac{d}{dx}\left\{[c^2 - V_A^2(x)]\frac{dV_x}{dx}\right\} - k^2[c^2 - V_A^2(x)]V_x = 0 \tag{8}$$

where

$$c \equiv \frac{\omega}{k}$$

The boundary conditions are taken as

$$x = x_1, x_2: \qquad V_x = 0 \tag{9}$$

We multiply equation (8) by the complex conjugate V_x^* of V_x, integrate from x_1 and x_2 and use the boundary conditions (9) to obtain

$$\int_{x_1}^{x_2} (c^2 - V_A^2)\left(\left|\frac{dV_x}{dx}\right|^2 + k^2|V_x|^2\right) dx = 0 \tag{10}$$

Writing $c = c_r + ic_i$, where c_r and c_i are the real and imaginary parts of c, and separating (10) into real and imaginary parts, we find

$$\int_{x_1}^{x_2} (c_r^2 - c_i^2 - V_A^2)\left(\left|\frac{dV_x}{dx}\right|^2 + k^2|V_x|^2\right) dx = 0 \tag{11}$$

$$2c_r c_i \int_{x_1}^{x_2} \left(\left|\frac{dV_x}{dx}\right|^2 + k^2|V_x|^2\right) dx = 0 \tag{12}$$

Equation (12) shows that, if the integrand is well behaved, one has

$$c_i = 0 \tag{13}$$

In view of this result, (11) implies that $(c_r^2 - V_A^2)$ must change sign for some x in (x_1, x_2) or that

$$c_r = \pm V_A \qquad \text{for some} \quad x \in (x_1, x_2) \tag{14}$$

We next introduce

$$W = c^2 - V_A^2, \qquad G = W^{1/2} V_x \tag{15}$$

so that equation (8) becomes

$$G'' - \left(k^2 + \frac{W''}{2W} - \frac{W'^2}{4W^2}\right) G = 0 \tag{16}$$

where the primes denote differentiation with respect to x. We multiply (16) by G^*, the complex conjugate of G, and integrate over x from x_1 and x_2, to obtain

$$\int_{x_1}^{x_2} (|G'|^2 + k^2|G|^2)\, dx + \int_{x_1}^{x_2} \frac{1}{4W^2}(2WW'' - W'^2)|G|^2\, dx = 0 \tag{17}$$

This is true if either of the following conditions is fulfilled:

(i) $G \equiv 0$ or $V_A \equiv c$ (18)

(ii) $G \not\equiv 0$ or $V_A \not\equiv c$ (19)

and

$$2WW'' - W'^2 < 0$$

Thus, discrete MHD modes exist when (18) or (19) is satisfied.

Away from the point $x = x_*$ where $c^2 = V_A^2$, if we assume that W varies slowly with x, then, as a first approximation, we may take W'' and W'^2 to be small, and an approximate solution of equation (16) is

$$G \approx e^{-k(x - x_*)}, \qquad x > x_* \tag{20}$$

Then, from (15), we have

$$V_x \approx \frac{1}{(c^2 - V_A^2)} e^{-k(x - x_*)}, \qquad x > x_* \tag{21}$$

which is the WKB approximation to the true solution of equation (8) and is valid only for $x \neq x_*$.

As an example, consider the case wherein $B_0 = \text{const}$. Then, equations (8) and (9) give

$$(c^2 - V_A^2)\left(\frac{d^2 V_x}{dx^2} - k^2 V_x\right) = 0 \tag{22a}$$

$$x = x_1, x_2: \qquad V_x = 0 \tag{22b}$$

This problem has two classes of solutions

(i) Discrete modes, which satisfy

$$\frac{d^2 V_x}{dx^2} - k^2 V_x = 0 \tag{23a}$$

$$x = x_1, x_2: \qquad V_x = 0 \tag{23b}$$

This class is empty for the problem (22). This result also follows due to the violation of (19) for the present case.

(ii) Continuous modes, which satisfy

$$c = \pm V_A \tag{24}$$

On the other hand, if $B_0(x)$ is continuous in an open interval I, then the interval I contributes to the spectrum those real values of the frequency ω such that $c = \pm V_A(x)$ for some $X \in I$.

If we have $c^2 = V_A^2$ for $x = x_*$, then in the neighborhood of $x = x_*$, equation (8) becomes

$$\eta \frac{d^2 V_x}{d\eta^2} + \frac{dV_x}{d\eta} - k^2 \eta V_x = 0 \tag{25}$$

where $\eta \equiv x - x_*$.

The solution of this equation

$$V_x = AI_0(k\eta) + BK_0(k\eta) \tag{26}$$

where A and B are constants and $I_0(z)$ and $K_0(z)$ are the modified Bessel functions of the first kind and the second kind, respectively. The power series expansion of (26) gives

$$V_x = A\left(1 + \frac{1}{4}k^2\eta^2 + \cdots\right) - B\left(\ln\frac{k\eta}{2} + \gamma\right)\left(1 + \frac{k}{4}k^2\eta^2 + \cdots\right)$$
$$+ B\left(\frac{1}{4}k^2\eta^2 + \cdots\right) \tag{27}$$

where γ represents Euler's constant.

The behavior of V_x as η passes from $\eta < 0$ to $\eta > 0$ can be determined by imagining that equation (8) has been obtained by a Laplace transform with $\mathrm{Im}(\omega) \ll \mathrm{Re}(\omega) \approx \omega$. In view of $\mathrm{Im}(x) = 0$, we find $\mathrm{Im}[k^2 V_A^2(x)] = 0$, $\mathrm{Im}(\partial k^2 V_A^2/\partial x)_* \approx 0$, and the imaginary part of

$$k^2 V_A^2(x) = \omega^2 + (x - x_*)\left(\frac{\partial k^2 V_A^2}{\partial x}\right)_*$$
$$= \omega^2\left(1 + \frac{x - x_*}{L} + \cdots\right) \tag{28}$$

leads to $\mathrm{Im}(x_*) \approx (2L/\omega)\,\mathrm{Im}(\omega) < 0$. As a result, η passes through $\eta = 0$, $\arg(\eta)$ changes from 0 for $\eta > 0$ to π for $\eta < 0$; $\ln \eta$ changes from $\ln|\eta|$ to $\ln|\eta| - i\pi$. The imaginary part in the latter expression represents the resonant absorption of the incident Alfven wave by the plasma. Alternatively, the x axis may be viewed as mapped onto a contour that passes below the real η axis in the complex η plane.

3. CONTINUOUS MHD SPECTRUM IN AN INCOMPRESSIBLE PLASMA

In order to study the continuous spectrum, let us Fourier transform V_x with respect to y,

$$V_x = \int_{-\infty}^{\infty} e^{-iky} V_x(x, y, t)\, dy \tag{29}$$

and formulate an initial-value problem for V_x with the boundary conditions (9):

$$\left(\frac{1}{k^2}\frac{\partial^2}{\partial t^2} + V_A^2\right)\left(\frac{\partial^2}{\partial x^2} - k^2\right) V_x + \left(\frac{dV_A^2}{dx}\right)\frac{\partial V_x}{\partial x} = 0 \tag{30a}$$

$$t = 0: \qquad V_x = V_0, \quad V_{x_t} = V_{t_0} \tag{30b}$$

We introduce the Laplace transform

$$\bar{V}_x(x, s) = \int_0^{\infty} e^{-st} V_x(x, t)\, dt \tag{31}$$

so that the system (30a), (30b) gives

$$\left(\frac{s^2}{k^2} + V_A^2\right)\left(\frac{d^2}{dx^2} - k^2\right)\bar{V}_x + \left(\frac{dV_A^2}{dx}\right)\frac{d\bar{V}_x}{dx} = g(x) \tag{32}$$

where $g(x)$ is given by

$$g(x) \equiv \frac{s}{k^2}\left(\frac{d^2 V_0}{dx^2} - k^2 V_0\right) + \frac{1}{k^2}\left(\frac{d^2 V_{t_0}}{dx^2} - k^2 V_{t_0}\right) \tag{33}$$

Using the Green's function method, we construct an explicit solution of (32) in the form

$$\bar{V}_x(x) = \left[\int_{-1}^{x} \bar{\psi}_1(\xi)\bar{\psi}_2(x) + \int_{x}^{1} \bar{\psi}_1(x)\bar{\psi}_2(\xi)\right] \frac{g(\xi)\, d\xi}{\{s^2/k^2 + V_A^2(\xi)\}\, \bar{W}(\bar{\psi}_1, \bar{\psi}_2)} \tag{34}$$

where

$$\bar{W}(\bar{\psi}_1, \bar{\psi}_2) \equiv \bar{\psi}_1(x)\bar{\psi}_2'(x) - \bar{\psi}_1'(x)\bar{\psi}_2(x) \tag{35}$$

and $\bar{\psi}_2$, $\bar{\psi}_2$ satisfy

$$\left[\left(\frac{s^2}{k^2}+V_A^2\right)\left(\frac{d^2}{dx^2}-k^2\right)+\left(\frac{dV_A^2}{dx}\right)\frac{d}{dx}\right]\bar{\psi}_{1,2}=0 \tag{36}$$

$$\bar{\psi}_1(-1)=0, \qquad \bar{\psi}_2(1)=0 \tag{37}$$

Here we have dimensionless distances using the width $2a$ of the domain, which is now taken to be $-1<x<1$.

We may construct $\bar{\psi}_1$, $\bar{\psi}_2$ from two linearly independent solutions $\bar{V}_{x_1}$, $\bar{V}_{x_2}$ of (32) as follows:

$$\bar{\psi}_1=\bar{V}_{x_1}(x)\bar{V}_{x_2}(-1)-\bar{V}_{x_1}(-1)\bar{V}_{x_2}(x) \tag{38}$$

$$\bar{\psi}_2=\bar{V}_{x_1}(x)\bar{V}_{x_2}(1)-\bar{V}_{x_1}(1)\bar{V}_{x_2}(x) \tag{39}$$

Thus,

$$\bar{W}(\bar{\psi}_1, \bar{\psi}_2)=-\bar{W}(\bar{V}_{x_1}, \bar{V}_{x_2})\cdot\Delta \tag{40}$$

where

$$\Delta\equiv\begin{vmatrix}\bar{V}_{x_1}(-1) & \bar{V}_{x_2}(-1)\\ \bar{V}_{x_1}(1) & \bar{V}_{x_2}(1)\end{vmatrix} \tag{41}$$

Finally, we obtain the solution for $V_x(x, t)$ by means of the inverse Laplace transformation

$$V_x(x, t)=\frac{1}{2\pi i}\int_{\propto-i\infty}^{\propto+i\infty}\bar{V}_x(x, s)\,e^{st}\,ds \tag{42}$$

It is noted that Green's function in (34) is a multiple-valued function of s with four logarithmic branch points on the imaginary axis of the complex s plane. These branch points are $\pm ikV_A(x)$, $\pm ikV_A(\xi)$, which also represent the poles of the integrand in (34). In order to evaluate (42), we have to cut the s plane in such a way that Green's function in (34) is single-valued in s. A suitable Bromwich contour may be drawn to evaluate the contributions to the solution from the branch points and the poles. If the s plane is cut along the intervals connecting the pairs of branch points, the numerator of the integrand in (34) becomes single-valued. The cuts associated with the numerator have to be combined with the cuts associated with the denominator to ensure the single-valuedness of the Green's function. There are also other poles of $\bar{V}_x(x, s)$, which arise at

$$\bar{W}(\bar{\psi}_1, \bar{\psi}_2)=0 \qquad \text{or} \quad \Delta=0 \tag{43}$$

This is simply the characteristic-value relation for the discrete spectrum of normal modes with $\text{Re}(s)=0$.

Thus, $V_x(x, t)$ can be expressed as

$$V_x(x, t) = \frac{1}{2\pi i} \int_{\propto - i\infty}^{\propto + i\propto} \bar{V}_x(x, s)\, e^{st}\, ds + \sum \text{exponentials like } e^{st} \tag{44}$$

where the summation is taken over the discrete spectrum. In the first term on the right-hand side in (44), contributions from the simple poles as $s = \pm ikV_A$ lead to terms like $\exp(\pm ikV_At)$, whereas the logarithmic singularities in $\bar{\psi}_1$ and $\bar{\psi}_2$ produce terms like $(1/t)\exp(\pm ikV_At)$ (Lighthill, 1964).

It is evident that continuous modes produced by a well-behaved initial perturbation will normally decay according to a power law.

We next show that some special singular solutions can, however, behave differently.

4. A SPECIAL SINGULAR MODE

Let us now take $B_0(x) = \text{const}$, and the initial conditions to be

$$t = 0: \qquad V_x = V_0(x, k), \qquad V_{x_t} = 0 \tag{45}$$

Equation (30a) then takes on the form

$$\left(\frac{\partial^2}{\partial t^2} + k^2V_A^2\right)\left(\frac{\partial^2}{\partial x^2} - k^2\right) V_x = 0 \tag{46}$$

Equation (46), in conjunction with (45), then gives the solution

$$V_x(x, k, t) = V_0(x, k) \cos kV_At \tag{47}$$

By inverting the Fourier transform with respect to y, we obtain the final solution

$$\begin{aligned} V_x(x, y, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iky} V_0(x, k) \cos kV_At\, dk \\ &= \tfrac{1}{2}[V_0(x, y + V_At) + V_0(x, y - V_At)] \end{aligned} \tag{48}$$

where $V_0(x, y)$ is the Fourier inverse of the transform $V_0(k, x)$. Equation (48) merely describes the propagation of a well-behaved initial perturbation with no damping. However, we will now show that some special singular solutions exist and do behave differently.

On taking the Laplace transform of (30a) and using (45), we obtain

$$x \gtrless 0: \qquad \bar{V}_x'' - k^2 \bar{V}_x = \frac{s(V_0'' - k^2 V_0)}{s^2 + k^2 V_A^2} \tag{49}$$

$$x = -1, 1: \qquad \bar{V}_x = 0 \tag{50}$$

The inverse Laplace transform of (49) gives for $x<0$ or $x>0$

$$V_x'' - k^2 V_x = -\tfrac{1}{2}\, e^{-kV_A t}(V_0'' - k^2 V_0) \tag{51}$$

As we saw previously with (23), there are no nonzero values of V_0 such that

$$V_0'' - k^2 V_0 = 0 \tag{52}$$

$$x = -1, 1: \qquad V_0 = 0 \tag{53}$$

so the right-hand side in (51) cannot vanish identically!

In order to solve equation (51) with the boundary conditions (50), we construct a Green's function $G(x, \xi)$ which satisfies the following:

$$\left(\frac{\partial^2}{\partial x^2} - k^2\right) G(x, \xi) = \delta(x - \xi) \tag{54}$$

$$x = -1, 1: \qquad G(x, \xi) = 0 \tag{55}$$

$$x = \xi: \qquad [G] = 0, \qquad \left[\frac{\partial G}{\partial x}\right] = 1 \tag{56}$$

where the square bracket denotes the jump of its contents.

One may find that

$$G(x, \xi) = \begin{cases} -\dfrac{1}{s} \sinh k(1+\xi) \sinh k(1-x), & \xi < x \\[2ex] -\dfrac{1}{s} \sinh k(1+x) \sinh k(1-\xi), & \xi > x \end{cases} \tag{57}$$

where

$$s = k \sinh k \tag{58}$$

Thus, the solution in terms of the Green's function has the form

$$V_x(k, x, t) = -\tfrac{1}{2} \int_{-1}^{1} G(x, \xi)\, e^{-ikV_A t}(V_0'' - k^2 V_0)\, d\xi \tag{59}$$

We consider the case where

$$V_0'' - k^2 V_0 = \delta(x - \tilde{\xi}) \tag{60}$$

so that (59) gives

$$V_x(k, x, t) = -\frac{1}{2s} e^{-ikV_A t} \times \begin{cases} \sinh k(1-x) \sinh k(1+\tilde{\xi}), & \tilde{\xi} < x \\ \sinh k(1+x) \sinh k(1-\tilde{\xi}), & \tilde{\xi} > x \end{cases} \tag{61}$$

Effecting the inverse Fourier transform with respect to y, we obtain

$$V_x(x, y, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iky} V_x(k, x, t)\, dk \tag{62}$$

We close the contour of integration in (62) along the real k axis by an infinite semicircle in the lower/upper half-plane for $y \lessgtr 0$, so that the contributions to $V_x(x, y, t)$ come from the poles at $k = \mp in\pi$, $n = 1, 2, 3, \ldots$.

Thus, we obtain (61) and (62); we have

$$V_x(x, y, t) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{e^{\pm n\pi y}}{2n\pi} \sin n\pi(1-x) \sin n\pi(1+\tilde{\xi})\, e^{-n\pi V_A t}, & \tilde{\xi} < x, \quad y \gtrless 0 \\ \displaystyle\sum_{n=1}^{\infty} \frac{e^{\pm n\pi y}}{2n\pi} \sin n\pi(1+x) \sin n\pi(1-\tilde{\xi})\, e^{-n\pi V_A t}, & \tilde{\xi} > x, \quad y \gtrless 0 \end{cases} \tag{63}$$

These results reveal that these modes decay exponentially but not as a power of t. Observe that this damping is not a consequence of the continuous variation of the field profile, because the field intensity is taken to be constant; it is rather a consequence of the phase mixing of the various Fourier components like the Landau damping in the plasma kinetic model. [The phase mixing is believed not to be a consequence of the continuous variation of the field profile (Cally, 1991).]

5. MHD SPECTRUM IN A RESISTIVE PLASMA

The MHD equations for an incompressible resistive plasma are given by (1)–(3) and

$$\frac{\partial B}{\partial t} = \nabla(\mathbf{V} \times \mathbf{B}) + \eta\, \nabla^2 \mathbf{B} \tag{64}$$

where η represents the resistivity of the plasma.

As before, we consider free oscillations. On linearizing the perturbations, we obtain from (1), (2), (4), and (64)

$$\nabla \cdot \left\{\left[\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} - \eta\, \nabla^2\right) - \frac{(B_0 \cdot \nabla)^2}{\rho}\right] \nabla V_x\right\} = 0 \tag{65}$$

On making a normal-mode analysis with the Fourier decomposition in y and t according to

$$V_x \sim V_x(x)\, e^{i(ky - \omega t)} \tag{66}$$

it turns out that V_x satisfies the equation

$$\frac{d}{dx}\left\{\left[k^2 V_A^2 - \omega^2 - i\omega\eta\left(\frac{d^2}{dx^2} - k^2\right)\right]\frac{dV_x}{dx}\right\}$$
$$-k^2\left[k^2 V_A^2 - \omega^2 - i\omega\eta\left(\frac{d^2}{dx^2} - k^2\right)\right] V_x = 0 \tag{67}$$

When the resistivity is small, it becomes important only near an Alfven resonance $k^2 V_A^2(x_*) = \omega^2$. Thus, in the neighborhood of the latter point, equation (67) can be approximated by

$$\frac{d^4 V_x}{d\xi^4} + \lambda^2\left(\xi \frac{d^2 V_x}{d\xi^2} + \frac{dV_x}{d\xi} + \xi V_x\right) = 0 \tag{68}$$

where

$$\xi \equiv ik(x - x_*), \qquad k^2 V_A^2 - \omega^2 = \left(\frac{dk^2 V_A^2}{dx}\right)_* (x - x_*)$$

and

$$\lambda^2 \equiv \left(\frac{dk^2 V_A^2}{dx}\right)_* \frac{1}{\omega\eta} \gg 1 \tag{69}$$

Equation (68) is similar to one which has been extensively studied by Wasow (1953) and Rabenstein (1958).

In order to construct explicit solutions to equation (68), let us make the Laplace transformation

$$\bar{V}_x(s) = \int_0^\infty e^{-s\xi} V_x(\xi)\, d\xi \tag{70}$$

Equation (68) then becomes

$$s^4\bar{V}_x-\lambda^2\left[\frac{d}{ds}(s^2\bar{V}_x+\bar{V}_x)\right]+\lambda^2 s\bar{V}_x=0 \tag{71}$$

which gives

$$\bar{V}_x(s)=\frac{A}{(s^2+1)^{1/2}}\exp\left(\frac{s^3}{3\lambda^2}-\frac{s}{\lambda^2}+\frac{1}{\lambda^2}\tan^{-1}s\right) \tag{72}$$

On inverting the Laplace transformation, we obtain

$$V_x(\xi)=\int_C e^{s\xi}\bar{V}_x(s)\,ds \tag{73}$$

where C represents a closed contour in the complex s plane.

For large s, the integrand in (73) is dominated by the term $\exp(s^3/3\lambda^2)$. The quantity $s^3/3\lambda^2$ has a negative real part in regions of the complex s plane, for which

$$\frac{2\pi n}{3}+\frac{\pi}{6}+\frac{2}{3}\arg\lambda<\arg s<\frac{\pi}{2}+\frac{2\pi n}{3}+\frac{2}{3}\arg\lambda \tag{74}$$

Because of the terms in the integrand involving powers of a complex number, those regions in (74) which differ by changes in n by a factor of 3 represent Riemannian sheets. Different choices of the contour C will produce different solutions $V_x(\xi)$, and we may find four different contours which produce four linearly independent solutions $V_x(\xi)$. However, each chosen contour represents a single solution valid for all ξ.

We next use the asymptotic method of steepest descent to evaluate the integral (73) for large $|\xi|$. The dominant contribution comes from the saddle points given by

$$\frac{df(s)}{ds}=0 \tag{75}$$

where $f(s)$ is the exponent involved in (73), given by

$$f(s)=\frac{s^3}{3\lambda^2}+s\left(\xi-\frac{1}{\lambda^2}\right)+\frac{1}{\lambda^2}\tan^{-1}s \tag{76}$$

Putting $s=\tan\theta$, the saddle points which are the solutions of (75) are given by the roots of the equation

$$\tan^4\theta+\lambda^2\xi\tan^2\theta+\lambda^2\xi=0 \tag{77}$$

For large $|\xi|$, (77) gives

$$\tan^2 \theta_{sp} \approx -\lambda^2 \xi + 1k \quad \text{or} \tag{78a}$$

$$-\left(1+\frac{1}{\lambda^2 \xi}\right) \tag{78b}$$

where θ_{sp} is the saddle point.

Thus, when $\tan^2 \theta_{sp} > 0$ (i.e., $\xi < 0$), the integration through the saddle point gives an evanescent contribution (or a growing mode). On the other hand, when $\tan^2 \theta_{sp} < 0$ (i.e., $\xi < 0$), the integration though the saddle point produces an oscillatory $\exp[\pm\frac{2}{3} i^{1/2} \lambda (x - x_*)^{3/2} k^{3/2}]$ variation. The saddle point given by (78b) always produces an oscillatory mode.

The above solutions are valid only in the neighborhood of the Alfven resonance layer $\omega = kV_A(x_*)$. In order to obtain the complete solutions, one has to match asymptotically these solutions to those valid outside the resonance layer—an issue not dealt with in this paper.

6. MHD SPECTRUM IN A COMPRESSIBLE PLASMA

The MHD equations for an ideal compressible plasma are

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{V} = 0 \tag{79}$$

$$\rho \frac{D\mathbf{V}}{Dt} = -\nabla p + (\nabla \times \mathbf{B}) \times \mathbf{B} \tag{80}$$

$$\frac{D}{Dt}\left(\frac{p}{\rho^\gamma}\right) = 0 \tag{81}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) \tag{82}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{83}$$

where γ is the ratio of specific heats of the plasma.

We consider free oscillations in a plasma subjected to a nonuniform magnetic field $\mathbf{B}_0 = B_0 \hat{\mathbf{i}}_y$ and in equilibrium given by

$$\frac{d}{dx}\left(p_0 + \frac{1}{2} B_0^2\right) = 0 \tag{84}$$

This magnetic field configuration, though apparently restrictive, is adequate for the resonance absorption of the slow mode, which requires $B_{0y} \neq 0$ [Hollweg and Yang (1988), who call it the "cusp resonance"]. Besides, this field configuration actually enables one to study a compressibility-affected Alfven resonance. In contrast, for the more general case with the equilibrium magnetic field along the y and z directions and the perturbations propagating along the y and z directions, the Alfven resonance turns out (Hasegawa and Uberoi, 1982) to be unaffected by compressibility.

We study linearized perturbations expressed in terms of normal modes like $\exp[i(ky-\omega t)]$. Introducing the Lagrangian variable

$$\mathbf{V}=\frac{\partial \xi}{\partial t} \tag{85}$$

it follows from (79)–(84) that

$$\rho\omega^2\xi_x=\frac{d\tilde{p}}{dx}-ikB_0B_x \tag{86}$$

$$\rho\omega^2\xi_y=ik\tilde{p}+B_x\frac{dB_0}{dx} \tag{87}$$

$$\tilde{p}=-\gamma p_0\left(ik\xi_y+\frac{d\xi_x}{dx}\right)-\xi_x\frac{dp_0}{dx} \tag{88}$$

$$B_y=-B_0\left(ik\xi_y+\frac{d\xi_x}{dx}\right)+ikB_0\xi_y-\xi_x\frac{dB_0}{dx} \tag{89}$$

where $\tilde{p}=p+B_0B_y$.

We obtain from equations (86)–(89) (Hasegawa and Uberoi, 1982)

$$\frac{d}{dx}\left[\rho_0\frac{c^2(1+V_A^2/a^2)-V_A^2}{1-c^2/a^2}\frac{d\xi_x}{dx}\right]-k^2\rho_0(c^2-V_A^2)\xi_x=0 \tag{90}$$

where a is the speed of sound in the plasma, $a\equiv(\gamma p_0/\rho_0)^{1/2}$, and $c=\omega/k$. A more general version of equation (90) was given by Goedbloed (1983). Observe that equation (90) reduces to equation (8) when the speed of sound a is very large [as Hasegawa and Uberoi (1982) pointed out], so that $c/a\ll 1$ and $V_A/a\ll 1$.[2]

[2]It is of interest to note that the inclusion of the thermal effects in the case of Alfven waves does not increase the order of the differential equation. This is in contrast with the case of longitudinal plasma waves, wherein the inclusion of the thermal effects enhances the order of the differential equation (Maggs and Morales, 1983). The latter is

$$V_T\frac{d^4\phi}{dx^4}+(\omega^2-\omega_p^2)\frac{d^2\phi}{dx^2}+\left[\frac{d}{dx}(\omega^2-\omega_p^2)\right]\frac{d\phi}{dx}-k^2(\omega^2-\omega_p^2)\phi=0$$

ϕ is the electrostatic potential, V_T is the thermal speed of electrons, and ω_p is the plasma frequency.

The boundary conditions are

$$x = x_1, x_2: \qquad \xi_x = 0 \tag{91}$$

It is noted that equation (91) is not valid when $c \approx a$. It is necessary to include the nonlinear terms for the case $c = a$.

We multiply equation (90) by the complex conjugate ξ_x^* and integrate over $x \in (x_1, x_2)$ and then use (91) to obtain

$$\int_{x_1}^{x_2} \left[\frac{c^2(1 + V_A^2/a^2) - V_A^2}{1 - c^2/a^2} \left| \frac{d\xi_x}{dx} \right|^2 + k^2(c^2 - V_A^2)|\xi_x|^2 \right] dx = 0 \tag{92}$$

Putting $c = c_r + ic_i$ and separating (92) into real and imaginary parts, we find

$$\int_{x_1}^{x_2} \rho_0 \left[\frac{[(c_r^2 - c_i^2)(1 + V_A^2/a^2) - V_A^2] \times [1 - (c_r^2 - c_i^2)/a^2] - (4c_r^2 c_i^2/a^2)(1 + V_A^2/a^2)}{1 - (c_r^2 - c_i^2)/a^2 + 4c_r^2 c_i^2/a^4} \times \left| \frac{d\xi_x}{dx} \right|^2 + k^2(c_r^2 - c_i^2 - V_A^2)|\xi_x|^2 \right] dx = 0 \tag{93}$$

$$2c_r c_i \int_{x_1}^{x_2} \rho_0 \left[\frac{|d\xi_x/dx|^2}{[1 - (c_r^2 - c_i^2)/a^2] + (4c_r^2 - c_i^2)/a^4} + k^2|\xi_x|^2 \right] dx = 0 \tag{94}$$

(94) shows that, if the integrand is well-behaved, we have

$$c_i = 0 \tag{95}$$

Using (95), we find that (93) becomes

$$\int_{x_1}^{x_2} \rho_0 \left[\frac{c_r^2(1 + V_A^2/a^2) - V_A^2}{1 - c_r^2/a^2} \left| \frac{d\xi_x}{dx} \right|^2 + k^2(c_r^2 - V_A^2)|\xi_x|^2 \right] dx = 0 \tag{96}$$

We have now two cases to consider.

(i) $a > c_r$ globally.

In this case, $\{c_r^2(1 + V_A^2/a^2) - V_A^2\}$ cannot be negative globally because $c_r^2 - V_A^2$ will also then be negative globally and so equation (96) cannot be satisfied. But $c_r^2(1 + V_A^2/a^2) - V_A^2$ can be positive globally if c_r lies globally in the window $V_A^2/(1 + V_A^2/a^2) < c_r^2 < V_A^2$, so that equation (96) can be satisfied. Thus, $c_r^2(1 + V_A^2/a^2) - V_A^2 \neq 0$ for all $x \in (x_1, x_2)$ if $V_A^2/(1 + V_A^2/a^2) < c_r^2 < V_A^2$ globally. However, $c_r^2(1 + V_A^2/a^2) - V_A^2 = 0$ for some $x \in (x_1, x_2)$ if the latter condition is violated. Note that the window $(V_A^2/(1 + V_A^2/a^2), V_A^2)$ will disappear in the limit $a \to \infty$, so that we revert to the known situation for the incompressible case, viz., that $c_r - V_A^2 = 0$ for some $x \in (x_1, x_2)$.

(ii) $a < c_r$ globally.

For this case, since

$$V_A^2 > V_A^2 - c_r^2 \tag{97}$$

we have

$$\frac{c_r^2}{a^2} V_A^2 > V_A^2 > V_A^2 - c_r^2$$

or

$$c_r\left(1 + \frac{V_A^2}{a^2}\right) - V_A^2 > 0, \qquad a < c_r \tag{98}$$

for all $x \in (x_1, x_2)$ so that $c_r^2(1 + V_A^2/a^2) - V_A^2 \neq 0$ for all $x \in (x_1, x_2)$. This implies that compressibility effects, if strong enough, can eliminate the slow-mode continuous spectrum just like the nonideal effects such as resistivity, though without causing an increase in the order of the differential equation.

The singularity where $c_r^2(1 + V_A^2/a^2) - V_A^2 = 0$ was also mentioned by Roberts (1981) and Hollweg and Yang (1988).

If we have $c = V_A^2/(1 + V_A^2/a^2)$ for $x = \hat{x}$, then in the neighborhood $x = \hat{x}$, equation (90) becomes

$$\eta \frac{d^2\xi_x}{d\eta^2} + \frac{d\xi_x}{d\eta} + \frac{k^2c^4}{\hat{a}(c^2 - a^2)} \xi_x = 0 \tag{99}$$

where

$$\hat{a} \equiv \left[\frac{d}{dx}(V_A^2)\right]_{x=\hat{x}}, \qquad \eta \equiv x - \hat{x}$$

Let us assume that $\hat{a} > 0$. The solution of equation (99) is

$$\xi_x = AI_0\left\{2kc^2\left[\frac{\eta}{\hat{a}(a^2 - c^2)}\right]^{1/2}\right\} + BK_0\left\{2kc^2\left[\frac{\eta}{\hat{a}(a^1 - c^2)}\right]^{1/2}\right\}, \qquad a^2 > c^2 \tag{100}$$

where A and B are constants. The power series expansion of (100) gives

$$\begin{aligned}\xi_x = A&\left[1 + \frac{k^2c^4}{\hat{a}(a^2 - c^2)}\eta + \cdots\right] \\ &- B\left\{\ln\left[kc^2\left(\frac{\eta}{a(a^2 - c^2)}\right)^{1/2}\right] + \gamma\right\}\left[1 + \frac{k^2c^4}{\hat{a}(a^2 - c^2)}\eta + \cdots\right] \\ &+ B\left[\frac{k^2c^4}{\hat{a}(a^2 - c^2)}\eta + \cdots\right], \qquad a^2 > c^2\end{aligned} \tag{101}$$

where γ is Euler's constant.

It turns out that even the discrete MHD spectrum in a compressible plasma behaves differently. Let us consider the case where $B_0(x)$, $\rho_0(x)$, and $p_0(x)$ jump from one set of constant values B_{01}, ρ_{01}, and p_{01} for $x<0$ to other constant values B_{02}, ρ_{02}, and p_{02} for $x>0$, and the speed of sound a is constant. In this case, we obtain from equation (90)

$$x \lessgtr 0: \qquad \frac{d^2\xi_x}{dx^2} - \frac{k^2(c^2 - V_{A1,2}^2)(1 - c^2/a^2)}{c^2(1 + V_{A1,2}^2/a^2) - V_{A1,2}^2}\,\xi_x = 0 \tag{102}$$

so that

$$x \lessgtr 0: \qquad \xi_{x1,2} = A_{1,2}\exp\left\{\pm kx\left[\frac{(c^2 - V_{A1,2}^2)(1 - c^2/a^2)}{c^2(1 + V_{A1,2}^2/a^2) - V_{A1,2}^2}\right]^{1/2}\right\} \tag{103}$$

The continuity of the normal velocity and the total pressure at the interface $x=0$ gives

$$x=0: \qquad [\xi_x]=0, \qquad \left[\frac{c^2(1 + V_A^2/a^2) - V_A^2}{1 - c^2/a^2}\frac{d\xi_x}{dx}\right] = 0 \tag{104}$$

where the square bracket denotes the jump of its contents.

Using (103), we find that (104) gives

$$c^2 = a^2 + \tfrac{1}{2}(V_{A1}^2 + V_{A2}^2) \pm [a^4 + (V_{A1}^2 + V_{A2}^2)^2]^{1/2} \tag{105}$$

In the limit $a \to \infty$, (105) reduces to the result for an incompressible plasma, namely, $c^2 = \frac{1}{2}(V_{A1}^2 + V_{A2}^2)$. However, (105) indicates that the discrete spectrum in a compressible plasma decays exponentially provided

$$a^2 < \tfrac{3}{4}(V_{A1}^2 + V_{A2}^2) \tag{106}$$

whereas its counterpart in an incompressible plasma is undamped. Therefore, the compressibility effects, when strong enough, appear to cause damping of the discrete modes. Physically, this is due to radiation to infinity of the energy associated with the perturbation by the sound waves.

7. CONCLUSIONS

The continuum MHD modes produced by a well-behaved initial perturbation are known to decay according to a power law. However, the analysis in Section 4 appears to show that some special singular solutions exist and behave differently. There seem to exist continuous modes which show exponential damping and are different from the ones resulting from a continuous variation of the magnetic field. Section 6 shows that discrete and continuous spectra behave differently in an essential way in a compressible plasma—whereas the discrete modes undergo an exponential damping even in an ideal plasma, the slow-mode continuous spectrum is eliminated if the compressibility effects are strong. The latter result is also brought about by the nonideal effects such as resistivity, though in a different manner, by increasing the order of the differential equation. Section 5 presents explicit solutions to the resistive MHD modes which are valid in the Alfven resonance layer at $\omega = kV(x_*)$.

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